

DIFFRACTION BY A STRIP

by

A. Erdélyi and C. H. Papas

CALIFORNIA INSTITUTE OF TECHNOLOGY

Pasadena, California.

A Technical Report to the Office of Naval Research

Prepared for

Office of Naval Research Contract

Nonr 220(14)

NR 071-262

November 1953

THIS REPORT HAS BEEN DELIMITED
AND CLEARED FOR PUBLIC RELEASE
UNDER DOD DIRECTIVE 5200.20 AND
NO RESTRICTIONS ARE IMPOSED UPON
ITS USE AND DISCLOSURE.

DISTRIBUTION STATEMENT A

APPROVED FOR PUBLIC RELEASE;
DISTRIBUTION UNLIMITED.

On Diffraction by a Strip

A. Erdelyi and C. H. Papas

Introduction

The problem of diffraction by an infinite strip or slit has been the subject of several investigations⁽¹⁾. There are at least two "exact" methods for attacking this problem. One of these is the integral equation method,⁽²⁾ the other the Fourier-Lamé method.⁽³⁾ The integral equation obtained for this problem cannot be solved in closed form; expansion of the solution in powers of the ratio (strip width/wavelength) leads to useful formulas for low frequencies. In the Fourier-Lamé method the wave equation is separated in coordinates of the elliptic cylinder, the solution appears as an infinite series of Mathieu functions, and the usefulness of the result is limited by the convergence of these infinite series, and by the available tabulation of Mathieu functions.

The variational technique developed by Levine and Schwinger avoids some of the difficulties of the above-mentioned methods and, at least in principle, is capable of furnishing good approximations for all frequency-ranges. The scattered field may be represented as the effect of the current induced in the strip, and it has been proved by Levine and Schwinger⁽⁴⁾ that it is possible to represent the amplitude of the far-zone scattered field in terms of the induced current in a form which is stationary with respect to small variations of the current about the true current. Substitution, in this representation, of a rough approximation for the current may give a remarkably good approximation of the far-zone scattered field amplitude. In this note we assume a normally incident field polarized parallel to the generators of the strip. As a rough approximation, we take a uniform density of the current induced in the strip. Since the incident magnetic field is constant over the strip, Fock's theory⁽⁵⁾ may be cited in support of the

uniformity of the current distribution, except near the edges where the behaviour of the field⁽⁶⁾ indicates an infinite current density. A more detailed analysis of the current, by Moullin and Phillips⁽³⁾, is available but was not used here.

Once the (approximate) amplitude of the far-zone field has been obtained, the scattering cross-section may be found by the application of the scattering theorem^(4,7,8) which relates this cross-section to the imaginary part of the amplitude of the far-zone scattered field along the central line of the umbral region. In spite of the crude approximation adopted for the induced current, the scattering cross-section shows a fair agreement with other available results.

Integral Equation

We assume a plane wave with harmonic time dependence $\exp(-i\omega t)$, normally incident on the perfectly conducting infinitesimally thin strip of width $2a$,

$$z = 0, \quad -a \leq y \leq a. \quad (1)$$

We further assume that the incident wave is polarized parallel to the edges of the strip (i.e. to the x axis) so that the only nonvanishing components of its electric field E , magnetic field H , and complex Poynting vector S (all measured in MKS units, bars denoting conjugate complex numbers) are

$$E_x^i = \exp(ikz) \quad (2)$$

$$H_y^i = k/(\omega\mu) \exp(ikz) \quad (3)$$

$$S_z^i = E_x^i \overline{H_y^i} = k/(\omega\mu) \quad (4)$$

The scattered electric field is again parallel to the x axis, and the total electric field is the sum of the incident and scattered field,

$$E_x(y,z) = E_x^i(y,z) + E_x^{sc}(y,z) \quad (5)$$

The boundary condition on the screen is the vanishing of the electric field

$$E_x(y, 0) = 0 \quad \text{for } -a < y < a; \quad (6)$$

and the scattered field, E_x^{sc} , must represent, at large distances from the screen, an outgoing cylindrical wave (Sommerfeld's radiation condition).

The expression of the total electric field in terms of the induced current $K_x(y)$ is

$$E_x(y, z) = \exp(ikz) - \omega\mu/4 \int_{-a}^a H_0^{(1)}(k[(y-y')^2 + z^2]^{1/2}) K_x(y') dy' \quad (7)$$

$i/4 H_0^{(1)}(k[(y-y')^2 + z^2]^{1/2})$ being the two-dimensional free-space Green's function. The scattered field in (7) certainly satisfies Sommerfeld's radiation condition; in order that it also satisfies (6), we must have

$$1 = \omega\mu/4 \int_{-a}^a H_0^{(1)}(k|y-y'|) K_x(y') dy' \quad |y| \leq a \quad (8)$$

and this is the integral equation of our problem.

Far-Zone Scattered Field Amplitude

We define the far-zone scattered field amplitude $A(\phi)$ by

$$E_x^{sc} \sim i/4 [2/(\pi i k \rho)]^{1/2} \exp(ik\rho) A(\phi) \quad k\rho \rightarrow \infty \quad (9)$$

where $\rho \cos \phi = z$ and $\rho \sin \phi = -y$; this expression represents a cylindrical outgoing wave of "amplitude" A .

Since

$$H_0^{(1)}(k[(y-y')^2 + z^2]^{1/2}) [2/(\pi i k \rho)]^{1/2} \exp(ik\rho + iky' \sin \phi) \quad (10)$$

when $k\rho \rightarrow \infty$, we have from (7), for large $k\rho$,

$$E_x^{sc} \sim i/4 [2/(\pi i k \rho)]^{1/2} \exp(ik\rho) \int_{-a}^a i\omega\mu \exp(iky' \sin \phi) K_x(y') dy' \quad (11)$$

Comparing (9) and (11) we get

$$A(\vartheta) = i\omega\mu \int_{-a}^a \exp(iky' \sin \vartheta) K_x(y') dy' \quad (12)$$

A Scattering Theorem

If P denotes the scattered energy flux per unit length, then the scattering cross-section σ is defined by

$$\sigma = P/S \quad (13)$$

where $S = 1/2 \operatorname{Re} S_z^1 = k/(2\omega\mu)$ is the incident energy flux per unit area.

$$P = 1/2 \operatorname{Re} \int_{-a}^a E_x^1(y'; 0) \bar{K}_x(y') dy' = 1/2 \operatorname{Re} \int_{-a}^a K_x(y') dy' \quad (14)$$

since $E_x^1 = 1$ at $z = 0$. Hence,

$$\sigma = \omega\mu/k \operatorname{Re} \int_{-a}^a K_x(y') dy' \quad (15)$$

But from (12)

$$A(0) = i\omega\mu \int_{-a}^a K_x(y') dy' \quad (16)$$

Therefore, comparing (15) and (16) we get

$$\sigma = \operatorname{Im} A(0)/k \quad (17)$$

It may be remarked that in the limit for very high frequencies, the cross-section (17) turns out to be twice the projected area. At first sight the factor two may seem inconsistent with geometrical optics, but actually the discrepancy is due to the different definitions of the scattering cross section.⁽⁹⁾

Variational Principle

We multiply the integral equation (8) by $K_x(y)$, integrate from $y = -a$

to $y = +a$, divide by the square of $\int_{-a}^a K_x(y) dy$, and then recall (16). Thus we obtain

$$A(0) = 4i \frac{\left[\int_{-a}^a K_x(y) dy \right]^2}{\int_{-a}^a \int_{-a}^a K_x(y) H_0^{(1)}(k|y-y'|) K_x(y') dy dy'} \quad (18)$$

In Appendix A it is shown that the expression (18) is stationary with respect to small variations of K_x about the true K_x which satisfies the integral equation (8). Consequently, if we substitute a reasonable trial function for the unknown K_x in (18), we expect to obtain a reasonably accurate value of $A(0)$.

Trial Function

According to Fock's theory⁽⁵⁾ the current induced in the central region of the strip is equal to twice the tangential component of the incident magnetic field. And according to Bouwkamp⁽⁶⁾ H_y must have a singularity like $(a^2 - y^2)^{-1/2}$ at the edges. This certainly conforms to the detailed information Moullin and Phillips⁽³⁾ reported. In the present paper we take for a trial function $K_x(y) = 1$, thus ignoring the effect of the edges.

Scattering Cross Section

With $K_x(y) = 1$, computation (see Appendix B) shows that (18) becomes

$$[A(0)]^{-1} = (4i a k)^{-1} \left\{ \int_0^{2ka} H_0^{(1)}(t) dt - H_1^{(1)}(2ka) + (\pi i k a)^{-1} \right\} \quad (19)$$

And applying the theorem (17) to (19), we get

$$\sigma'/(4a) = \frac{\int_0^{2ka} J_0(t) dt - J_1(2ka)}{\left[\int_0^{2ka} J_0(t) dt - J_1(2ka) \right]^2 + \left[\int_0^{2ka} Y_0(t) dt - Y_1(2ka) - (\pi ka)^{-1} \right]^2} \quad (20)$$

The integrals appearing here have been tabulated. (10)

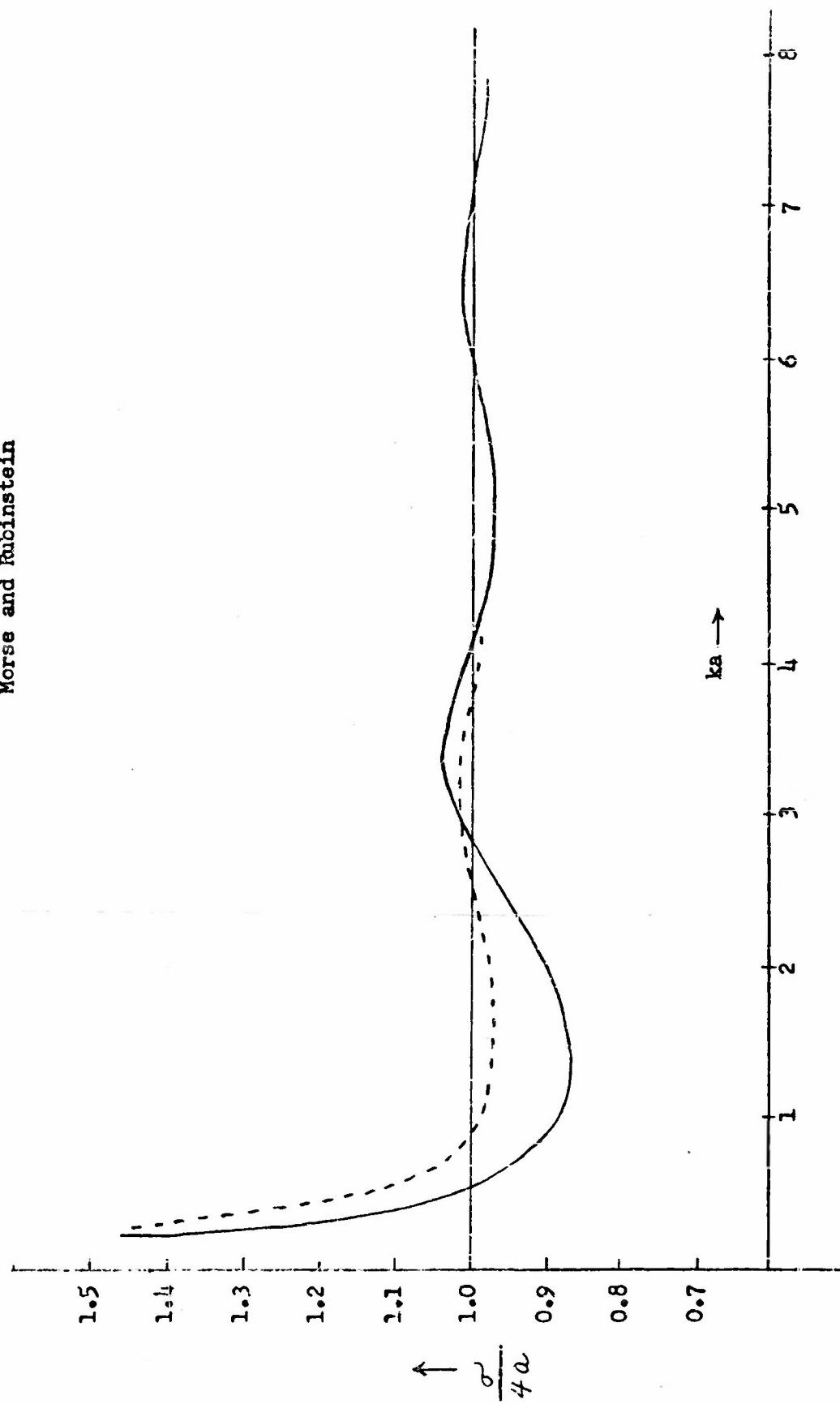
Results

A plot of $\sigma'/(4a)$ versus ka is shown in the graph. This is a plot of eq.(20). For small ka we have Rayleigh scattering. As ka increases the curve performs a damped oscillation about $\sigma'/(4a) = 1$. And for $ka \rightarrow \infty$ it can be shown by means of asymptotic representations of Bessel functions and their integrals that $\sigma' \rightarrow 4a$.

According to Babinet's principle⁽¹¹⁾ the problem we have discussed is complementary to the problem of scattering by a slit for a normally incident wave polarized perpendicular to the axis of the slit. For intermediate values of ka our curve behaves qualitatively as Morse and Rubinstein's⁽¹⁾: the quantitative agreement is not very good, the deviation being due to our choice of an overly simplified trial function. It is quite remarkable how such a rough approximation of the induced current yields fairly good results over the entire spectrum.

Variational method, trial function constant, eg. 20

Morse and Rubinstein



REFERENCES

- (1) B. Sieger, Ann. d. Physik 27, 622 (1908)
 W. Wien, Jahresbericht d. deutsch Math. Verein 15, 42 (1906)
 K. Schwarzschild, Math. Ann. 55, 177 (1902)
 P. M. Morse and P. J. Rubinstein, Phys. Rev. 54, 895 (1938)
 Lord Rayleigh, Phil. Mag. (5) 43, 259 (1897); 44, 28 (1897)
 P. S. Epstein, Encyklopadie für Mathematische Wissenschaften, Vol. 5 III

- (2) J. W. Miles, J. of Math. and Phys., 28, 223 (1950)
 Groschwitz u. H. Hönl, Z. Physik 131, 305 (1952)
 A. Sommerfeld, Vorlesung über theoretische Physik, Bd. IV Optik, 39.
 Rolf Müller u. Konrad Westpfahl, Z. für Physik, 134, 245 (1953)

- (3) E. B. Moullin and F. M. Phillips, Proc. Inst. Elec. Engrs. Part IV, 22
137 (1952, Morse and Rubinstein l.c.(1))

- (4) H. Levine and J. Schwinger, Comm. Pure and App. Math. 3, 355 (1950).
 See also B. B. Baker and E. T. Copson, the Mathematical Theory of
 Huyghen's Principle, second ed., Oxford (1950) Chapter V § 6.2; N.
 Marcuvitz in the New York University Seminar Notes, Recent Develop-
 ments in the Theory of Wave Propagation, section III D, New York
 (1953); and A. Erdélyi, Acc. d. Sci. Torino, Atti, 87 (1952-53) for
 the scalar case.

- (5) V. Fock, J. of Phys. of U.S.S.R., 10, 130 (1946)

- (6) C. J. Bouwkamp, Physica, 12, 467 (1946; J. Meixner, Ann.d.Phys.(6)
6 2 (1949)

- (7) C. H. Papas, J. App. Phys., 21, 318 (1950)

- (8) V. Twersky, Research Report No. EM-54, Math. Res. Group, N.Y.U.

- (9) L. Brillouin, J. App. Phys. 20, 1110 (1949)

- (10) Lowan and Abramowitz, J. Math. Phys. 22, 2 (1943)
 G. N. Watson, Bessel Functions, Cambridge (1945)

- (11) P. S. Epstein, l.c.(1); B. B. Baker and E. T. Copson, l.c.(4).

Appendix A

We show here that $A(0)$ of (18) is stationary with respect to small variations of $K_X(y)$ about the true $K_X(y)$ which satisfies the integral equation (8).

From (18) and the calculus of variations it follows that

$$\begin{aligned} \delta A & \int_{-a}^a \int_{-a}^a K_X(y) H_0^{(1)}(k|y-y'|) K_X(y') dy dy' \\ & + A \int_{-a}^a \int_{-a}^a \delta K_X(y) H_0^{(1)}(k|y-y'|) K_X(y') dy dy' \\ & + A \int_{-a}^a \int_{-a}^a K_X(y) H_0^{(1)}(k|y-y'|) \delta K_X(y') dy dy' = \\ & 2 \int_{-a}^a \delta K_X(y) dy \int_{-a}^a K_X(y) dy . \end{aligned}$$

Rearranging terms we get

$$\begin{aligned} \delta A & \int_{-a}^a K_X(y) H_0^{(1)}(k|y-y'|) K_X(y') dy dy' = \\ & \int_{-a}^a \delta K_X(y) dy \left\{ \int_{-a}^a K_X(y) dy - A \int_{-a}^a H_0^{(1)}(k|y-y'|) K_X(y') dy' \right\} \\ & + \int_{-a}^a \delta K_X(y') dy' \left\{ \int_{-a}^a K_X(y) dy - A \int_{-a}^a H_0^{(1)}(k|y-y'|) K_X(y) dy \right\} . \end{aligned}$$

The right side of this equation disappears for small arbitrary variations $\delta K_X(y)$ because the quantities in curly brackets are zero by virtue of (12), i.e., $A(0) = i\omega\mu \int_{-a}^a K_X(y) dy$, and the integral equation (8). Consequently $\delta A = 0$ and the expression (18) is stationary.

Appendix B

For trial function $K_x(y) = \text{constant}$, (16) becomes

$$\frac{1}{A(0)} = \frac{1}{16 \pi a^2} \int_{-a}^a \int_{-a}^a H_0^{(1)}(k|y-y'|) dy dy'$$

Put $k(y+y') = s$, $k(y-y') = t$. Then

$$\frac{1}{A(0)} = \frac{1}{32 \pi (ka)^2} \iint_S H_0^{(1)}(|t|) ds dt$$

where S is the square with vertices $(\pm 2ka, 0)$, $(0, \pm 2ka)$.

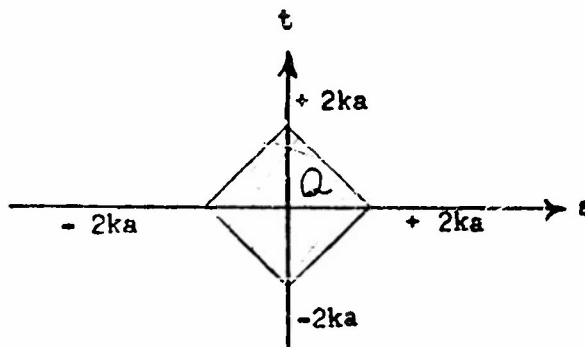
Let Q be the quadrant $s > 0, t > 0, s + t \leq 2ka$. Then

$$\frac{1}{A(0)} = \frac{1}{8 \pi (ka)^2} \iint_Q H_0^{(1)}(t) dt = \frac{1}{8 \pi (ka)^2} \int_0^{2ka} (2ka - t) H_0^{(1)}(t) dt.$$

Since $\int_0^{2ka} t H_0^{(1)}(t) dt = 2ka H_1^{(1)}(2ka) + \frac{2i}{\pi}$, the above equation becomes

$$\frac{1}{A(0)} = \frac{1}{4 \pi ka} \left\{ \int_0^{2ka} H_0^{(1)}(t) dt - H_1^{(1)}(2ka) + \frac{1}{\pi ka} \right\}.$$

And this is (19).



CALIFORNIA INSTITUTE OF TECHNOLOGY

Distribution List

Technical Reports

<u>Addressee</u>	<u>Attn</u>	<u>No. of Copies</u>
Chief of Naval Research	Code 427	2
Navy Department	460	2
Washington 25, D. C.		
Director	Code 2000	6
Naval Research Laboratory	3480	1
Washington 25, D. C.		
Commanding Officer		1
Office of Naval Research Branch Office		
1000 Geary Street		
San Francisco 9, California		
Commanding Officer		1
Office of Naval Research Branch Office		
1030 E. Green Street		
Pasadena, California		
Commanding Officer		1
Office of Naval Research Branch Office		
The John Crerar Library Building		
86 East Randolph Street		
Chicago 1, Illinois		
Commanding Officer		1
Office of Naval Research Branch Office		
346 Broadway		
New York 13, New York		
Officer-in-Charge		2
Office of Naval Research		
Navy No. 100		
Fleet Post Office		
New York, New York		
Chief, Bureau of Ordnance	Re 4	1
Navy Department		
Washington 25, D. C.		
Chief, Bureau of Ships	Code 810	2
Navy Department		
Washington 25, D. C.		

<u>Addressee</u>	<u>Attn</u>	<u>No. of Copies</u>
Chief, Bureau of Aeronautics Navy Department Washington 25, D. C.	EL-51	1
Chief of Naval Operations Navy Department Washington 25, D. C.	Op-413 Op-20X Op-32	1 1 1
Director Naval Ordnance Laboratory White Oak, Maryland		1
Commander U. S. Naval Electronics Laboratory San Diego, California		1
Commander Naval Air Development Center Johnsville, Pennsylvania	AAEL	1
Commanding Officer and Director U.S. Navy Underwater Sound Laboratory New London, Connecticut		1
U.S. Naval Post Graduate School Monterey, California	Librarian	1
U.S. Coast Guard 1300 E. Street, N.W. Washington 25, D. C.	EEE	1
Assistant Secretary of Defense (Research and Development) Research and Development Board Department of Defense Washington 25, D. C.		1
Armed Services Technical Information Agency Document Service Center Knott Building Dayton 2, Ohio		5
Office of Technical Services Department of Commerce Washington 25, D. C.		1
Director National Bureau of Standards Connecticut Ave. and Van Ness St, N.W. Washington 25, D. C.	Dr. N. Smith Div.14, Librarian	1 1

<u>Addressee</u>	<u>Attn</u>	<u>No. of Copies</u>
Electrical Engineering Department University of California Berkeley, California	Dr. L. Reukema	1
Department of Electrical Engineering Cornell University Ithaca, New York	Dr. C. R. Burrows	1
Department of Electrical Engineering University of Illinois Urbana, Illinois	Prof. E. C. Jordan	1
Applied Physics Laboratory Johns Hopkins University 8621 Georgia Avenue Silver Spring, Maryland		1
Radiation Laboratory Johns Hopkins University 1315 St. Paul Street Baltimore 2, Maryland		1
Research Laboratory of Electronics Document Room Massachusetts Institute of Technology Cambridge 39, Massachusetts	Mr. J. Hewitt	1
Stanford Research Institute Stanford, California	Dr. John V.N. Granger	1
Electrical Engineering Department University of Texas Box F, University Station Austin, Texas		1
Bell Telephone Laboratories, Inc. Murray Hill, New Jersey		1
General Electric Company The Knolls Schenectady, New York	Mr. R. Damon	1
Radio Corporation of America RCA Laboratories Division Princeton, New Jersey	Librarian	1
Watson Laboratories Library, AMC Red Bank, New Jersey	ENAGS1	1
Electronics Research Laboratory Stanford University Stanford, California	Dr. F. E. Terman	1

<u>Addressee</u>	<u>Attn</u>	<u>No. of Copies</u>
Mathematics Research Group New York University 45 Astor Place New York, New York	Dr. M. Kline	1
Chairman Department of Engineering University of California Los Angeles, California		1
Harvard University Cambridge, Massachusetts	Dr. R. King	1
Director Los Alamos Scientific Laboratory Los Alamos, New Mexico		1